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STATIONARY ISOTOPIES OF INFINITE-DIMENSIONAL SPACES

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Abstract. Let X denote the Hilbert cube or any separable infinite-dimensional Fréchet space. It has been shown that any two homeomorphisms f, g of X onto itself is isotopic to each other by means of an invertible-isotopy on X. In this paper we generalize the above results to the extent that if f, g are K-coincident on X (that is, $f(x)=g(x)$ for $x \in K$), then the isotopy can be chosen to be K-stationary provided K is compact and has property-Z in X. The main tool of this paper is the Stable Homeomorphism Extension Theorem which generalizes results of Klee and Anderson.

1. An *invertible-isotopy* of space X to space Y is a homeomorphism F of $X \times I$ onto $Y \times I$ such that $F(X \times t) = Y \times t$ for all $t \in I = [0, 1]$. We denote such an F by $\{f_t = F|_{X \times t}\}_{t \in I}$. For $K \subset X$, F is *K-stationary* if for each $x \in K$, $f_t(x) = f_0(x)$ for all t. The main results of this paper are Theorem 1.1 and Theorem 2.1.

THEOREM 1.1. *If X is the Hilbert cube or s (on any separable infinite-dimensional Fréchet space) and K is a compact set with property-Z in X, then any two K-coincident homeomorphisms f, g of X onto X are isotopic by an invertible-isotopy F such that F is of K-stationary.*

Two maps $f_1, f_2 \colon X \to Y$ are *A-coincident*, $A \subset X$, if $f_1(x) = f_2(x)$ for all $x \in A$. A closed set A of X has *property*-Z in X (following Anderson [2]) if for each homotopically trivial nonnull open subset U of X, $U \backslash A$ is both nonnull and homotopically trivial.

The Hilbert cube Q is the infinite product space $\prod_{i=1}^{\infty} J_i$ and s is the space $\prod_{i=1}^{\infty} \operatorname{Int} J_i$, where $J_i = J = [-1, 1]$. s is homeomorphic to any separable infinite-dimensional Fréchet space [0]. We consider s as imbedded canonically in Q, s is sometimes called the pseudointerior (PI) of Q. Let $B(Q) = Q \backslash s$. A homeomorphism of Q onto itself is a β^*-homeomorphism (following Anderson [2]) if $h(s) = s$. Let $G(X)$ denote the group of all homeomorphisms of X onto itself and let $G^*(Q)$ denote all β^*-homeomorphisms of $G(Q)$. A set K in Q is *Core* if K has the form $\prod_{i=1}^{\infty} [-a_i, a_i]$ where each $0 < a_i < 1$ and $[-a_i, a_i] \subset \operatorname{Int} J_i$. Theorem 1.1 follows from the following.

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THEOREM 1.2. *Let $X=Q$ or s and let K be a compact set with property-Z in X. If h is a homeomorphism of X onto X such that $h|_K=$ identity, then there is an invertible-isotopy $\{h_t\}$ of X onto itself such that $h_1=h$, $h_0=$ identity and $h_t|_K=$ identity for all t.*

Proof of Theorem 1.1. Let f, g be given as in Theorem 1.1. Consider $h=f^{-1}g$. Then $h|_K=$identity. Let $\{h_t\}$ be given by Theorem 1.2. Then $\{f\cdot h_t\}$ is an invertible-isotopy between f and g.

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2. Stable Homeomorphism Extension Theorem.

THEOREM 2.1. *Let $X=Q$ or s and let K be a compact set in $X\times I$ such that $K\cap X\times t\neq\varnothing$ for all t and for $X=Q$, $K\subset s\times I\subset Q\times I$. Let h be a homeomorphism of K into $X\times I$ such that* (1) *$h(K\cap X\times t)\subset X\times t$ for all t,* (2) *$h|_{K\cap X\times\{0,1\}}=$ identity and* (3) *for $X=Q$, $h(K)\subset s\times I$.*

Then there is an $h_1\in G(X\times I)$ such that

(a) *$h_1(X\times t)=X\times t$ for all t,*

(b) *$h_1|_K=h$ and*

(c) *$h_1|_{X\times\{0,1\}}=$ identity.*

Following the notations of Theorem 2.1, we have

COROLLARY 2.2. *If X, K and h are given as above satisfying* (1), (3) *and* (2'): *$h|_{K\cap X\times 1}=f|_{K\cap X\times 1}$ and $h|_{K\cap X\times 0}=f'|_{K\cap X\times 0}$ where f and f' are given homeomorphisms on $X\times 1$ and $X\times 0$ respectively, and when $X=Q$, f and f' are β^*-homeomorphisms. Then there is an $h_1\in G(X\times I)$ such that h_1 satisfies* (a), (b), *and*

(c') *$h_1|_{X\times 1}=f$ and $h_1|_{X\times 0}=f'$.*

Proof. By [7] there is an invertible-isotopy $F=\{f_t\}$ of X onto itself such that $f_1=f$, $f_0=f'$ and for $X=Q$, each f_t is β^*. Consider $h'=F^{-1}h$. h' satisfies the hypothesis of Theorem 2.1 and let h_1' be given by the theorem for h'. Then $h_1=Fh_1'$ is the desired function.

COROLLARY 2.3. *If X, K and h are given as above satisfying* (1), (3) *of the hypothesis, then there is an $h_1\in G(X\times I)$ satisfying* (a) *and* (b).

(This corollary can also be proved directly from the procedures used in [1].)

Proof. By [1, Theorem 4.2] there are homeomorphisms f, f' of X onto itself such that $f|_{K\cap X\times 1}=h|_{K\cap(X\times 1)}$ and $f'|_{K\cap X\times 0}=h|_{K\cap X\times 0}$ and for $X=Q$, f and f' are β^*-homeomorphisms. Now apply Corollary 2.2.

QUESTION 1. *Can the compact set K be replaced by a closed set with property-Z in Theorem* 1.1?

As it stands Theorem 2.1 cannot be generalized to an arbitrary closed Z-set in $s\times I$. A simple example would be to consider $K=C\times\{0\}\subset s\times J$ where C is a wild

0-dimensional closed set in s [8]. It is not difficult to see that K is a closed Z-set in $s \times J$. But a level-preserving imbedding of K into $s \times J$ that pushes C inside $s \times J$ obviously has no extension. A more meaningful question would be: In Theorem 2.1, can K be replaced by a closed set which has property-Z at each level? The answer to this question is yes. In a subsequent paper the author gives a solution to the question which appeared as a special case of a more general theorem.

Theorem 2.1 may be regarded as a stable generalization of homeomorphism extension theorems of V. Klee [5] for $X = l_2$ and of Anderson [1], [2]. Before we prove Theorem 1.2, we need the following.

3. Lemma 3.1. *Let h be any homeomorphism of Q onto Q such that for some compact subset K of s, $h(K) = K$. Then there is a sub-Hilbert cube $K' \subset s$ such that $K \subset K'$ and $h(K') \subset s$.*

Proof. Let π_i denote the projection mapping of Q into the ith coordinate. Write $Q \backslash s$ as $K_1 \cup K_2 \cup \cdots$ where each $K_i = \pi_j^{-1}(-1)$ or $\pi_j^{-1}(1)$ for some j. Let G denote the space of all imbeddings of Q into Q under the usual supremum metric. G is completable [6, p. 32]. Let G' be the subset of G consisting of all $f \in G$ such that $f(K) = K$. It is easy to see that G' is closed in G. Hence is of 2nd-category. Let $G_i = \{f' \in G' \mid f'(Q) \cap (K_i \cup h(K_i)) = \varnothing\}$. We claim that each G_i (namely G_1) is both open and dense in G'. G_1 is clearly open. To show G_1 is dense, suppose $g \in G'$ and $\varepsilon > 0$. Consider $h^{-1}g(Q)$, $h^{-1}g(K) = K$. Since $K \cap K_1 = \varnothing$, by moving the endpoints of the first coordinate interval inside a little bit, it is clear that there is a $g_1 \in G'$ such that $g_1(h^{-1}g(Q)) \cap K_1 = \varnothing$. Hence $h[g_1 h^{-1} g(Q)] \cap h(K_1) = \varnothing$. Similarly there is a $g_2 \in G'$ such that $g_2[hg_1h^{-1}g(Q)] \cap (K_1 \cup h(K_1)) = \varnothing$. Since g_1, g_2 can be chosen arbitrarily small, we can get $f' = (g_2)(hg_1h^{-1})g$ arbitrarily close to g or $d(f', g) < \varepsilon$. This shows G_1 is dense and the proof of the claim is complete. Now let $f \in \bigcap_{i \geq 1} G_i$. Hence $f(Q) \subset s$ and $f(Q) \cap h(Q \backslash s) = \varnothing$. Let $K' = h^{-1}(f(Q))$. K' has the desired properties.

Alternative proof. Note that $A = B(I^\infty) \cup h^{-1}(B(I^\infty))$ misses K. $Q \backslash A \cong s$ [3, Theorem II]. Thus there is a sub-Hilbert cube K' of $Q \backslash A$ containing K. K' is also a sub-Hilbert cube of Q, $K' \subset s$ and $h(K') \subset s$.

Proposition 3.2 (Anderson [2]). *For $X = Q$ or s, any homeomorphism between two closed sets each with property-Z in X can be extended to a homeomorphism of X onto X. Furthermore, if $X = Q$ and both closed sets are contained in s, then the extension can be chosen to be a β^*-homeomorphism.*

Proposition 3.3 (Wong [7]). *For $X = Q$ or s, any homeomorphism f of X onto itself is isotopic to the identity mapping by means of an invertible-isotopy $\{f_t\}$ satisfying the following:*

(1) *if $X = Q$ and if $Q' \subset Q$ is Core in Q such that $f(Q') \subset s$, then we can require $f_t(Q') \subset s$ for all t and*

(2) *if $X = Q$ and f is a β^*-homeomorphism, then each f_t is a β^*-homeomorphism.*

Note that properties (1) and (2) are not stated specifically in [7] but follow immediately from the construction.

4. Proof of Theorem 1.2.

Case 1. $X=Q$. By Proposition 3.2, there is an $f\in G(Q)$ such that $f(K)\subset s$. Let $g=fhf^{-1}$ and let $K_1=f(K)$. $g|_{K_1}=$identity. By Lemma 3.1 there is a sub-Hilbert cube $Q_1\subset s$ such that $K_1\subset Q_1$ and $g(Q_1)\subset s$. Then for some subset Q' of Q which is Core in Q, $Q_1\subset Q_1'$.

By Proposition 3.2 there is a β^*-homeomorphism $f_1\in G(Q)$ such that $f_1(Q_1)=Q_1'$. Let $g_1=f_1gf_1^{-1}$. Since f_1 is β^*, $g_1(Q_1')\subset s$. By Proposition 3.3 there is an invertible-isotopy $\{g_t\}_{t\in I}$ between g_1 and $g_0=$identity such that $g_t(Q_1')\subset s$ for all t. Let $K'=f_1(K_1)$, $K'\subset Q_1'\subset s$, $g_1|_{K'}=$identity. Let G be the homeomorphism of $Q\times I$ onto $Q\times I$ defined by $G|_{Q\times t}=g_t$. Then $G(K'\times t)\subset s\times t$ for all t. Thus $G'=G|_{K'\times I}$ satisfies the conditions that $G'(K'\times t)\subset s\times t$ and $G'|_{K'\times\{0,1\}}=$identity. By Theorem 2.1, there is a homeomorphism ψ of $Q\times I$ onto itself such that (1) $\psi(Q\times t)=Q\times t$ for all t, (2) $\psi|_{K'\times I}=G'$ and (3) $\psi|_{Q\times\{0,1\}}=$identity. Denote ψ by $\{\psi_t\}$ and denote $\psi_t^{-1}g_t$ by Φ_t. Then (1) $\Phi_t|_{K'}=$identity for all t and (2) $\Phi_1=g_1$, $\Phi_0=$identity. Let $h_t=(f_1f)^{-1}\Phi_tf_1f$. $\{h_t\}$ is the desired isotopy.

Case 2. $X=s$. By Proposition 3.3 there is an invertible-isotopy $G=\{g_t\}$ of s onto itself such that $g_1=h$, $g_0=$identity. By Theorem 2.1 there is a homeomorphism F of $s\times I$ onto itself satisfying (1), (2), and (3) of Theorem 2.1. Denote $F|_{s\times t}$ by f_t and denote $f_t^{-1}g_t$ by h_t. Then $\{h_t\}$ is the desired isotopy.

5. An example. The following example shows that the requirement of K in Theorem 1.1 to have property-Z in Q is sometimes necessary.

Write $Q=J_1\times J_2\times\cdots$ where each $J_i=[-1,1]=J$. Let K be the set $\{(x_1,x_2,\ldots)\in Q \mid x_1=0\}$. K does not satisfy property-Z in Q. Let $f\in G(Q)$ be the identity mapping and let $g\in G(Q)$ be defined by $g(x_1,x_2,\ldots)=(-x_1,x_2,x_3,\ldots)$. f, g are K-coincident. But there exists no invertible-isotopy between f and g which is also K-stationary. To see this let us suppose the contrary. Let $F=\{f_t\}$ be such an isotopy. Let $P=(-1,0,0,\ldots)\in Q$ and let $\pi_1\colon Q\to J_1$ be the projection. Then $\pi_1(F(P\times I))=[-1,1]$. So for some t, $0=\pi_1F(P\times t)=\pi_1f_t(P)$ contradict to the assumption that $f_t|_K=$identity. The following corollary clarifies the situation.

COROLLARY 5.1. *Let $X=Q$ and let f, g and K be as above. Suppose $h\in G(X)$ such that g, h are K-coincident, then either there is an invertible-isotopy between f and h which is K-stationary or there is an invertible-isotopy between g and h which is K-stationary.*

Proof. Let $X_1=\{x\in Q \mid x_1\leqq 0\}$, $X_2=\{x\in Q \mid x_1\geqq 0\}$. $K=X_1\cap X_2$. Since $h|_K=$identity, then either $h(X_1)=X_1$ or $h(X_1)=X_2$.

Case 1. $h(X_1)=X_1$. Let $h_1=h|_{X_1}$ and $h_2=h|_{X_2}$. For $i=1,2$, K has property-Z in X_i. By Theorem 1.1 there is an invertible-isotopy $H_i=\{h_t^i\}$ of X_i onto itself between h_i and the identity such that H_i is K-stationary. Define $H=\{h_t\}$ by $h_t|_{X_i}=h_t^i$.

Case 2. $h(X_1)=X_2$. Then gh satisfies the assumption of Case 1. Let $H=\{h_t\}_{t\in I}$ be defined as in Case 1 for gh. Then $h'_t=\{gh_t\}$ is the desired isotopy.

6. Throughout this section, let N be the set of positive integers and we write $Q=\prod_{i\in N} J_i$. Let π_i denote the projection of Q onto J_i. If $\alpha_1, \alpha_2\subset N$, $\alpha_1 \cup \alpha_2=N$, $\alpha_1 \cap \alpha_2=\varnothing$ and both α_1, α_2 are infinite, then we say Q factors into $Q_1\times Q_2$ where $Q_1=\prod_{i\in\alpha_1} J_i$ and $Q_2=\prod_{i\in\alpha_2} J_i$. Suppose $\alpha\subset N$ and $\{X_i\}_{i\geq 1}$ are spaces. Let $h\in G(\prod_{i\in\alpha} X_i)$, then h' is a natural extension of h onto $\prod_{i\in N} X_i$ if $h'(x, y)=(h(x), y)$ for all $x\in\prod_{i\in\alpha} J_i$ and $y\in\prod_{i\notin\alpha} J_i$. We consider Q as a convex subset of the linear space $\prod_{i=1}^{\infty} R_i$, $R_i=$reals.

Proof of Theorem 2.1. The theorem is explicitly shown for $X=Q$. For $X=s$, note that when restricted to $s\times I$ the constructed homeomorphisms are the desired ones. A proof is given for the special case in which Q factors into $Q_1\times Q_2$, and $K\cup h(K)\subset Q_1\times 0\times I$. The proof is completed by reducing the general case to the special one.

Consider the following for the special case. Let $\varphi(t)=t(1-t)$. $Q_i=J_{i1}\times J_{i2}\times\cdots$. Let $e\colon Q_1\to Q_2$ be the identification so that $\pi_{2j}e(x)=\pi_{ij}(x)$. Let $H(J)$ be the space of homeomorphisms of the interval $J=[-1, 1]$ leaving the endpoints fixed. $H(J)\cong s$ [4]. For $a, b\in \operatorname{Int}(J)$, let $p\omega(a, b, J)$ be the simple piecewise linear homeomorphism of J taking $[-1, a]$ to $[-1, b]$ and $[a, 1]$ to $[b, 1]$. For $x\in Q$ and $t\in I$, $\varphi(t)\cdot x$ means coordinate multiplication.

Let $\bar{g}_i\colon Q_1\times I\to H(J_{2i})$ be the map where $\bar{g}_i(x, t)=p\omega(0, \varphi(t)\cdot\pi_{1i}(x), J_{2i})$. Define g a homeomorphism of $Q_1\times\prod_{i>0} J_{2i}\times I$ by

$$g(x_1, x_{21}, x_{22},\dots, t) = (x_1, \bar{g}_1(x_1, t)(x_{21}), \bar{g}_2(x_1, t)(x_{22}),\dots, t).$$

Note that this homeomorphism is the identity for $t=0, 1$, and the point $(x, 0, t)$ in K is taken to $(x, \varphi(t)\cdot e(x), t)$.

Let $\tilde{k}_i\colon h(K)\cup Q_1\times 0\times\{0, 1\}\to H(J_{2i})$ be defined as

$$\tilde{k}_i(x, 0, t) = p\omega(0, \varphi(t)\cdot\pi_{1i}h^{-1}(x, 0, t), J_{2i}).$$

Let $\tilde{k}$ having the same domain and going into $\prod_{i>0} H(J_{2i})$ be defined by the coordinate maps $\tilde{k}_i$. By Dugundji's theorem $\tilde{k}$ can be extended to all of $Q_1\times 0\times I$. Call the extension $\bar{k}$. Define k a homeomorphism of $Q_1\times Q_2\times I$ by

$$k(x_1, x_2, t) = (x_1, k(x_1, 0, t)(x_2), t).$$

Note that this homeomorphism is the identity for $t=0, 1$, and the point $h(x, 0, t)$ in $h(K)$ is taken to $(\pi_1 h(x, 0, t), \varphi(f)\cdot e(x), t)$.

Let K' be the projection of $g(K)$ into $0\times Q_2\times I$, which is the same as the projection of $kh(K)$. For $t\neq 0, 1$, these projections are one-to-one. Let $x=e^{-1}(y)/\varphi(t)$; the point $(0, y, t)$ in K' corresponds to (x, y, t) in $g(K)$ and $(\pi_1 h(x, 0, t), y, t)$ in $kh(K)$.

Let $\tilde{f}_i\colon K'\cup 0\times Q_2\times\{0, 1\}\to H(J_{1i})$ be the map where

$$\tilde{f}_i(0, y, t) = p\omega(\pi_{1i}x, \pi_{1i}h(x, 0, t), J_{1i}),$$

if $t \neq 0, 1$ and $x = e^{-1}(y)/\varphi(t)$; and $\bar{f}_i(0, y, t) =$ identity on J_{1i} if $t = 0, 1$. As in defining $\bar{k}$, define $\bar{f}: 0 \times Q_2 \times I \to \prod_{i>0} H(J_{1i})$. Define f a homeomorphism of $Q_1 \times Q_2 \times I$ by

$$f(x_1, x_2 t) = (\bar{f}(0, x_2, t)(x_1), x_2, t).$$

Note that this homeomorphism is the identity for $t = 0, 1$ and the point $(x, \varphi(t) \cdot e(x), t)$ in $g(K)$ goes to the point $(\pi_1 h(x, 0, t), e(x) \cdot \varphi(t), t)$ in $kh(K)$.

The desired homeomorphism is $h_1 = k^{-1} fg$. h_1 is level-preserving and the identity at $t = 0, 1$. $h_1|_K = h$ as follows. Let $(x, 0, t) \in K$ and $t \neq 0, 1$. $h_1(x, 0, t) = k^{-1} fg(x, 0, t) = k^{-1} f(x, \varphi(f) \cdot e(x), t) = k^{-1}(\pi_1 h(x, 0, t), \varphi(f) \cdot e(x), t) = h(x, 0, t)$.

In the general case with no condition on $K \cup h(K)$, there is a β^*-homeomorphism l on $Q \times I$, where $Q = Q_1 \times Q_2$, such that $l(K \cup h(K)) \subseteq Q_1 \times 0 \times I$, and l is independent of the I coordinate [2]. $lhl^{-1}: l(K) \to lh(K)$ satisfies the special case. If h_1 extends lhl^{-1} then the desired extension is $l^{-1} h_1 l$.

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